

CRITICAL VALUES OF MOMENT MAPS ON QUANTIZABLE MANIFOLDS

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ABSTRACT. Let M be a quantizable symplectic manifold acted on by $T = (S^1)^r$ in a Hamiltonian fashion and J a moment map for this action. Suppose that the set M^T of fixed points is discrete and denote by $\alpha_{pj} \in \mathbb{Z}^r$ the weights of the isotropy representation at p . By means of the α_{pj} 's we define a partition $\mathcal{Q}_+, \mathcal{Q}_-$ of M^T . (When $r = 1$, \mathcal{Q}_\pm will be the set of fixed points such that the half of the Morse index of J at them is even (odd)). We prove the existence of a map $\pi_\pm : \mathcal{Q}_\pm \rightarrow \mathcal{Q}_\mp$ such that $J(q) - J(\pi_\pm(q)) \in I_\mp$, for all $q \in \mathcal{Q}_\pm$, where I_\pm is the lattice generated by the α_{pj} 's with $p \in \mathcal{Q}_\pm$. We define partition functions N_p similar to the ones of Kostant [7] and we prove that $\sum_{p \in \mathcal{Q}_+} N_p(l) = \sum_{p \in \mathcal{Q}_-} N_p(l)$, for any $l \in \mathbb{Z}^r$ with $|l|$ sufficiently large.

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1. INTRODUCTION

Let (M, ω) be a closed oriented symplectic manifold of dimension $2n$. Henceforth we assume that $\frac{1}{2\pi}[\omega]$ belongs to the image of the map $H^2(M, \mathbb{Z}) \rightarrow H^2(M, \mathbb{R})$; in other words, (M, ω) is *quantizable* [21]. Let (L, ∇) be prequantum data on M , that is, L is a Hermitian line bundle on M and ∇ is a connection on L whose curvature is $-i\omega$.

Let T be the torus $(U(1))^r$ and we assume that T acts on M in a Hamiltonian fashion. We denote by μ

$$\mu : M \rightarrow \mathfrak{t}^* = \overbrace{i\mathbb{R} \oplus \cdots \oplus i\mathbb{R}}^r$$

the corresponding normalized moment map (that is, we suppose that $\int_M \langle \mu, X \rangle \omega^n = 0$, for all $X \in \mathfrak{t}$). We will write J for the \mathbb{R}^r valued map $-i\mu$.

Throughout this Section we assume that the *prequantum data are T -invariant* (see [10] and Section 2), and that the set of fixed points M^T is a set of isolated points. If $p \in M^T$, we denote by $\alpha_{pj} \in \mathbb{Z}^r$, $j = 1, \dots, n$ the weights of the isotropy representation R of T on the tangent space to M at p .

Now we restrict ourselves to the case $r = 1$. Given $p \in M^{U(1)}$, we set

$$\mathcal{A}_p := \{i \mid \alpha_{pi} > 0\}, \quad \mathcal{B}_p := \{k \mid \alpha_{pk} < 0\}, \quad \sigma(p) := (-1)^{\#\mathcal{B}_p}.$$

We put $b := \#\mathcal{B}_p$ and $a := n - b$. Given a natural number l , we denote by $N_p(l)$ the following cardinal

$$(1.1) \quad \#\left\{ (m_1, \dots, m_a, n_1, \dots, n_b) \mid J(p) + \sum_{i \in \mathcal{A}_p} m_i \alpha_{pi} - \sum_{k \in \mathcal{B}_p} n_k \alpha_{pk} = l, m_i \in \mathbb{N}_{>0}, n_k \in \mathbb{N} \right\}.$$

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That is, $N_p(l)$ is the number of times we can write l as a sum of the type specified in (1.1). If $a = 0$, N_p is the Kostant partition function associated with the representation R^* , but in general our “partition” functions do not agree with the ones of Kostant.

On the other hand, one has the partition $M^{U(1)} = \mathcal{Q}_+ \cup \mathcal{Q}_-$, where

$$\mathcal{Q}_\pm = \{p \in M^{U(1)} \mid \sigma(p) = \pm 1\}.$$

In Subsection 3.1 we will prove the following Theorem

Theorem 1. *For $l \in \mathbb{N}$ sufficiently large*

$$\sum_{p \in \mathcal{Q}_+} N_p(l) = \sum_{p \in \mathcal{Q}_-} N_p(l).$$

Let us denote by I_\pm the ideal of \mathbb{Z} generated by the set

$$\{\alpha_{qj} \mid j = 1, \dots, n; q \in \mathcal{Q}_\pm\}.$$

In Subsection 3.1 the following Theorem is proved

Theorem 2. *Given $p_\pm \in \mathcal{Q}_\pm$, there is $q_\mp \in \mathcal{Q}_\mp$ such that*

$$(1.2) \quad J(p_\pm) - J(q_\mp) \in I_\mp.$$

There is a natural number c_\pm , $1 \leq c_\pm \leq \sharp \mathcal{Q}_\mp$ such that

$$(1.3) \quad c_\pm I_\pm \subset I_\mp.$$

In a neighborhood of the fixed point p for the circle action there are suitable coordinates $x_1, \dots, x_n, y_1, \dots, y_n$, such that the moment map J in this neighbourhood is given by

$$(1.4) \quad J(p) + \frac{1}{2} \sum_{j=1}^n \alpha_{pj} (x_j^2 + y_j^2).$$

Thus Theorem 2 gives a relation between the values of the moment map J at the fixed points and the Hessian of J , relative to those coordinates, at these points.

By (1.4) the index of the critical point p of the Morse function J is $2(\sharp \mathcal{B}_p)$. We can classify the points $p \in M^{U(1)}$ in even and odd according to the parity of $(1/2)\text{index}(p)$. Thus p is even iff $p \in \mathcal{Q}_+$. Hence we have the following Corollary, in whose statement we put $I_{\text{odd}} := I_-$.

Corollary 3. *For each even critical point p of J there is an odd critical point q such that*

$$J(p) - J(q) \in I_{\text{odd}}.$$

Obviously, the statement obtained from Corollary 3 exchanging “even” for “odd” is also true.

In this note we will prove two types of properties for $(U(1))^r$ -Hamiltonian actions:

(A) Properties with a combinatorial content (as the statement of Theorem 1), in which are involved the weights of the isotropy representations.

(B) Relations satisfied by the difference of critical values of the moment map, as the result stated in Theorem 2.

Convexity properties of a moment map μ for a Hamiltonian action of a torus T on a symplectic manifold were studied by Atiyah in [1] and by Guillemin and

Sternberg in [9]. The basic result they proved is that $\text{im } \mu$ is a convex polytope in \mathfrak{t}^* ; it is, in fact, the convex hull of the points $\mu(p)$, where p is a fixed point of the T -action. This result was generalized by Kirwan in [12] to Hamiltonian actions of compact connected Lie groups. Furthermore, Theorem 3 of [9] describes the “shape of the vertex” $\mu(p)$ of the polytope $\text{im } \mu$ in terms of the weights of the isotropy representation of T on the tangent space to M at p .

From the convexity properties of the moment map for actions on quantizable manifolds one deduces:

1.) The difference between two critical values of J belongs to integer lattice of \mathbb{R}^r (see [9] and [20]). We will obtain this result again in Section 2 when we explain briefly the concept of invariant prequantum data (see (2.2)).

2.) If $J(p)$ and $J(q)$ are vertices of $\text{im } J$ which determine an edge of $\text{im } J$, the local convexity theorem of [9] implies that $J(p) - J(q)$ belongs to the lattice generated by $\{\alpha_{qj}\}_j$.

In this paper not all the fixed points play identical role. They will be classified in two classes, say \mathcal{Q}_+ , \mathcal{Q}_- , using the weights of the respective isotropy representations. The properties of type **A** give the equality between the sum of our partitions functions for the points of \mathcal{Q}_+ and the corresponding one for the points of \mathcal{Q}_- . The proof we will give is elementary and independent of the known formulas for the Kostant partition functions [4], [8].

Our properties of type **B** give differences between critical values of J belonging to different class. So Corollary 3, Theorem 2 and, in general, the properties of type **B** give restrictions to some differences between critical values of the moment map, which can not be deduced from the above convexity properties.

The proofs of our results are based in the equivariant index formula for the Dirac operator [13], [3]. Although this formula has been used in different articles to prove “that quantization commutes with reduction” (see for example [5], [6], [11], [15], [18] or the research report [17]), in Section 2 we review this formula and write it in a form which will be convenient for our applications.

The case of circle actions with only isolated fixed points is considered in Subsection 3.1. We will prove Theorem 1 and Proposition 4; Theorem 2 is a straightforward consequence of Proposition 4.

The case when M is acted on by $T = (U(1))^r$ and M^T is a discrete set is considered in Subsection 3.2. In this case a way to define a partition of M^T in two subsets is by means of a polarizing vector (i.e. a vector non-orthogonal to α_{pj} , for all p and j). The corresponding properties of type **A** and type **B** are the statements of Theorem 5 and Theorem 7, respectively. We also define other more general partitions of M^T which will give rise to Theorem 8 and Theorem 9, whose statements are properties of type **A** and type **B**, respectively. These theorems are the generalizations of Theorem 1 and Theorem 2 to the case in which r is arbitrary. As a corollary we will prove that there is no an open half space of \mathbb{R}^r containing all the weights α_{pj} . Although the theorems of Subsection 3.1 are particular cases of the corresponding ones of Subsection 3.2, for the sake of clarity of the exposition we give a direct proof of them.

Finally, in Section 4 we analyze the case when M^T is a non necessarily discrete set. The statements of Theorem 10 and Corollary 11 are a properties of type **A**. This theorem in turn generalizes Theorem 5. We will apply Theorem 10 to a

particular case and we will obtain simple combinatorial relations satisfied by the weights of the isotropy representations (see the first two items of Proposition 12).

It is not easy to deduce general simple properties of type **B** when the fixed points are not isolated. We consider this point in a particular case in the Example at the end of the paper. The result is the third item of Proposition 12.

2. EQUIVARIANT INDEX THEOREM

Given a loop $\{\phi_t\}_{t \in [0,1]}$ in the Hamiltonian group $\text{Ham}(M, \omega)$ at the identity (see [14], [16]), let Y_t be the time-dependent vector field defined by

$$\frac{d\phi_t}{dt} = Y_t \circ \phi_t.$$

If f_t is the corresponding normalized Hamiltonian ($df_t = -\iota_{Y_t}\omega$), we can pose the problem of determining a family $\{\sigma_t\}$ of sections of L satisfying

$$\frac{d\sigma_t}{dt} = -\nabla_{Y_t}\sigma_t - if_t\sigma_t, \quad \sigma_0 = \sigma,$$

where $\sigma \in \Gamma(M, L)$ is a given section. It turns out that $\sigma_1 = \kappa(\phi)\sigma$, where the constant $\kappa(\phi)$ is given by

$$\kappa(\phi) = \exp\left(i \int_S \omega - i \int_0^1 f_t(p) dt\right),$$

p being an arbitrary point of M and S an arbitrary 2-cycle in M whose boundary is the curve $\{\phi_t(p)\}_t$ (see [19]). If p is a fixed point, i.e. $\phi_t(p) = p$ for all t , then

$$(2.1) \quad \kappa(\phi) = \exp\left(-i \int_0^1 f_t(p) dt\right).$$

Suppose that the torus $T = (U(1))^r$ acts on M in a Hamiltonian fashion. As we said, by μ will be denoted the corresponding normalized moment map. Given $X \in \mathfrak{t}$, we adopt the following convention

$$d\langle \mu, X \rangle = \omega(X_M, \cdot),$$

where X_M is the vector field on M defined by

$$X_M(p) = \frac{d}{d\epsilon}(e^{-\epsilon X} \cdot p)|_{\epsilon=0}.$$

For each $X \in \mathfrak{t}$ one can consider the operator

$$\mathcal{P}_X := \nabla_{X_M} + i\langle \mu, X \rangle : \Gamma(M, L) \rightarrow \Gamma(M, L).$$

\mathcal{P} is a representation of the Lie algebra \mathfrak{t} . The prequantum data are said to be T -invariant if the representation \mathcal{P} can be lifted to a representation of T [10].

From now on we assume that *the prequantum data (L, ∇) are T -invariant*. If X belongs to the integer lattice Λ of \mathfrak{t} ($\Lambda = \ker(\exp : \mathfrak{t} \rightarrow T)$) and p is a fixed point for the T -action, by applying (2.1) to $\phi_t(q) = e^{tX} \cdot q$ one obtains

$$(2.2) \quad \langle \mu(p), X \rangle \in 2\pi\mathbb{Z}.$$

Given an almost complex structure \mathcal{I} on M compatible with ω , we can consider the spaces $\Omega^{i,j}(M, L)$ of L -valued forms of type (i, j) . By means of the Hermitian connection on the canonical bundle on M we can define the spin- \mathbb{C} Dirac operator (see [13])

$$D_{\mathbb{C}} : \Omega^{0,\text{even}}(M, L) \rightarrow \Omega^{0,\text{odd}}(M, L),$$

and the corresponding virtual vector space

$$\text{ind}(D_{\mathbb{C}}) := [\ker(D_{\mathbb{C}})] - [\text{coker}(D_{\mathbb{C}})].$$

Since two almost complex structures on M compatible with ω are homotopic, $\text{ind}(D_{\mathbb{C}})$ is independent of \mathcal{I} . If \mathcal{I} is T -invariant, there is a natural representation of T in $\text{ind}(D_{\mathbb{C}})$.

We will study the character χ of this representation of T . If $t \in T$, $\chi(t)$ is a Laurent polynomial, that is, a finite sum of the form

$$(2.3) \quad \chi(t) = \sum_{\text{finite sum}} E(\rho)t^\rho,$$

where $\rho \in \mathbb{Z}^r$ and $E(\rho) \in \mathbb{Z}$.

One defines the moment μ^L as the T -equivariant form with values in $\text{End}(L)$ given by the expression (see [3])

$$\mu^L(X) = \mathcal{P}_X - \nabla_{X_M},$$

\mathcal{P}_X being the above infinitesimal action on $\Gamma(M, L)$. The equivariant curvature of the connection ∇ is $R_T(X) = R + i\mu^L(X)$, where R is the curvature of ∇ . So $R_T(X) = -i\omega - \langle \mu, X \rangle$ and this is a T -equivariant closed form; i.e.,

$$(d_T R_T)(X) := d(R_T(X)) + i\iota(X_M)(R_T(X)) = 0.$$

The T -equivariant Chern character of L , $\text{Ch}_T(L, X)$ is defined as the equivariant cohomology class of

$$(2.4) \quad \exp(iR_T(X)) = \exp(\omega - i\langle \mu, X \rangle).$$

Similarly we can define $\text{Td}_T(M, X)$, the T -equivariant Todd class of M .

For $X \in \mathfrak{t}$ sufficiently close to zero so that M_0 , the zero set of X_M , is $\{p \in M : e^X \cdot p = p\}$, the equivariant index formula gives the following expression for $\chi(e^X)$

$$(2.5) \quad \chi(e^X) = \int_M \text{Ch}_T(L, X) \text{Td}_T(M, X).$$

The localization formula in equivariant cohomology gives (see [2], [3])

$$(2.6) \quad \chi(e^X) = \int_{M_0} \frac{\text{Ch}_T(L, X) \text{Td}_T(M, X)}{e_T(\mathcal{N}, X)},$$

where $e_T(\mathcal{N}, X)$ is the equivariant Euler class of the normal bundle \mathcal{N} to M_0 in M [3].

Now we assume that the T -action has only isolated points. If M_0 is equal to $M^T = \{p \in M \mid t \cdot p = p, \text{ for all } t \in T\}$ it follows from (2.4) together with (2.6) the following expression for the character χ

$$(2.7) \quad \chi(t = e^X) = \sum_{p \in M^T} \frac{t^{J(p)}}{\prod_{j=1}^n (1 - t^{-\alpha_{pj}})},$$

where $\alpha_{pj} = (\alpha_{pj}^1, \dots, \alpha_{pj}^r) \in \mathbb{Z}^r$, $j = 1, \dots, n$ are the weights of the T -action on $T_p M$, and

$$t^{J(p)} = e^{-i\langle \mu(p), X \rangle}.$$

3. ISOLATED CRITICAL POINTS.

Throughout this Section we will assume that M^T is a discrete set.

3.1. Case $T=U(1)$. Now let us suppose that $r = 1$. Then for a generic $z \in U(1)$, as $M^{U(1)}$ is a discrete set

$$(3.1) \quad \chi(z) = \sum_{p \in M^{U(1)}} \frac{z^{J(p)}}{\prod_{j=1}^n (1 - z^{-\alpha_{pj}})}.$$

By (2.2) $J(p) \in \mathbb{Z}$, and the right hand side of (3.1) extends to a meromorphic function on $\mathbb{C} \cup \{\infty\}$.

Proof of Theorem 1.

If $0 < |z| < 1$ and $a \in \mathbb{Z}$

$$(3.2) \quad \frac{1}{1 - z^{-a}} = \begin{cases} -\sum_{m=0}^{\infty} z^{(m+1)a}, & \text{if } a > 0 \\ \sum_{m=0}^{\infty} z^{-ma}, & \text{if } a < 0 \end{cases}$$

Therefore, if $0 < |z| < 1$ the righthand side of (3.1) can be written

$$(3.3) \quad (-1)^n \sum_{p \in M^{U(1)}} \sigma(p) z^{J(p)} \left(\prod_{i \in \mathcal{A}_p} \sum_{m=1}^{\infty} z^{m\alpha_{pi}} \right) \left(\prod_{k \in \mathcal{B}_p} \sum_{m=0}^{\infty} z^{-m\alpha_{pk}} \right).$$

So an arbitrary summand of (3.3) has the following form

$$(3.4) \quad (-1)^n \sigma(p) z^{J(p) + \sum_i m_i \alpha_{pi} - \sum_k n_k \alpha_{pk}},$$

with $n_k \in \mathbb{N}$ and $m_i \in \mathbb{N}_{>0}$. The number of times that the monomial $+(-1)^n z^l$ appears in (3.3) is precisely

$$\sum_{p \in \mathcal{Q}_+} N_p(l),$$

and similarly for the monomial $-(-1)^n z^l$. Since $\chi(z)$ is a *finite* sum of monomials in the variable z , all except a finite number of terms of the form (3.4) with $\sigma(p) = +1$, must cancel all the terms with $\sigma(p) = -1$ except a finite number of them. That is, for $l \in \mathbb{N}$ large enough $\sum_{p \in \mathcal{Q}_+} N_p(l) = \sum_{p \in \mathcal{Q}_-} N_p(l)$. \square

Example 1 (S^1 -action on $\mathbb{C}P^2$).

Now we consider the symplectic toric manifold M associated to the Delzant polytope Δ in $(\mathbb{R}^2)^*$ with vertices $P(0,0)$, $Q(1,0)$ and $R(0,1)$. By J_T we denote the moment map of the \mathbb{T}^2 -action. Let us consider the S^1 -action generated by $X = (x, y) \in \mathbb{Z}^2$ through the \mathbb{T}^2 -action. The corresponding moment map is $J = \langle J_T, X \rangle$. If $x, y > 0$ and $x > y$, then $M^{S^1} = \{p, q, r\}$, where $p = J_T^{-1}(P)$, $q = J_T^{-1}(Q)$, $r = J_T^{-1}(R)$; moreover $J(p) = 0$, $J(q) = x$ and $J(r) = y$. The weights of the isotropy representations of $U(1)$ are

$$\alpha_{p1} = x, \alpha_{p2} = y; \alpha_{q1} = -x, \alpha_{q2} = -x + y; \alpha_{r1} = x - y, \alpha_{r2} = -y.$$

Thus $\mathcal{A}_p = \{1, 2\}$, $\mathcal{A}_q = \emptyset$, $\mathcal{A}_r = \{1\}$, $\mathcal{Q}_+ = \{p, q\}$ and $\mathcal{Q}_- = \{r\}$.

Given $l \in \mathbb{N}$ sufficiently large, we set

$$\mathcal{S}_p := \{(m_1, m_2) \mid l = m_1x + m_2y\}, \quad \mathcal{S}_q := \{(n_1, n_2) \mid l = x + n_1x + n_2(x - y)\}$$

$$\mathcal{S}_r := \{(m, n) \mid l = y + m(x - y) + ny\},$$

where $m, m_1, m_2 \in \mathbb{N}_{>0}$ and $n, n_1, n_2 \in \mathbb{N}$. The map $\mathcal{S}_r \rightarrow \mathcal{S}_p \coprod \mathcal{S}_q$ defined by

$$(m, n) \mapsto \begin{cases} (m, 1 - m + n) \in \mathcal{S}_p, & \text{if } 1 - m + n > 0 \\ (n, -1 + m - n) \in \mathcal{S}_q, & \text{if } 1 - m + n \leq 0. \end{cases}$$

is bijective. As $N_b(l) = \sharp \mathcal{S}_b$, for $b \in \{p, q, r\}$, it follows $N_r(l) = N_p(l) + N_q(l)$, which is the assertion of Theorem 1 in this particular case.

Proposition 4. *Given $p \in \mathcal{Q}_+$, there is $q(p) \in \mathcal{Q}_-$ such that*

$$J(p) - J(q(p)) \in I_-.$$

Given $i \in \mathcal{A}_p$, there is a natural number c , $1 \leq c \leq \sharp \mathcal{Q}_-$ such that $c\alpha_{pi} \in I_-$.

Proof.

As (3.3) is a finite sum, given $p \in \mathcal{Q}_+$, $\tilde{i} \in \mathcal{A}_p$ and $m \in \mathbb{N}$ sufficiently large there exist $q \in \mathcal{Q}_-$, m'_i, n'_k such that

$$J(p) + m\alpha_{p\tilde{i}} = J(q) + \sum_i m'_i \alpha_{qi} - \sum_k n'_k \alpha_{qk}.$$

So

$$(3.5) \quad J(p) - J(q) + m\alpha_{p\tilde{i}} \in I_-.$$

If we repeat the argument for $m+1$ we will conclude that there is $q_1 \in \mathcal{Q}_-$ such that $J(p) - J(q_1) + (m+1)\alpha_{p\tilde{i}} \in I_-$, etc. Since the number of points of \mathcal{Q}_- is finite, there is a point in \mathcal{Q}_- , which we denote by \hat{q} , and an integer c , $1 \leq c \leq \sharp \mathcal{Q}_-$ that

$$J(p) - J(\hat{q}) + m\alpha_{p\tilde{i}} \in I_- \text{ and } J(p) - J(\hat{q}) + (m+c)\alpha_{p\tilde{i}} \in I_-,$$

for some m large enough. Thus

$$(3.6) \quad c\alpha_{p\tilde{i}} \in I_-.$$

Reasoning with the element $c\alpha_{p\tilde{i}}$ as before with $\alpha_{p\tilde{i}}$, we can conclude that there is an element $q(p) \in \mathcal{Q}_-$ such that

$$J(p) - J(q(p)) + m_1 c \alpha_{p\tilde{i}} \in I_-,$$

with m_1 a natural number sufficiently large. From (3.6) it follows that

$$J(p) - J(q(p)) \in I_-.$$

□

Proof of Theorem 2. The arguments used in the proof of Proposition 4 applied to the case that $p \in \mathcal{Q}_-$, together with Proposition 4 complete the proof of Theorem 2. □

3.2. Case $T = (U(1))^r$. Let $t = (t_1, \dots, t_r)$ be a generic element of T sufficiently close to the identity. As we are assuming that T -action has only isolated points, then $\chi(t)$ can be calculated by (2.7). The right hand side of (2.7) is defined on the region of \mathbb{C}^r consisting of the points t such that $t^{\alpha_{pj}} \neq 1$ for any $p \in M^T$ and any j .

As $\alpha_{pj} \neq 0$, for a generic vector $u \in \mathbb{R}^r$

$$\alpha_{pj}(u) := \sum_{e=1}^r u_e \alpha_{pj}^e \neq 0,$$

for all $p \in M^T$ and all $j = 1, \dots, n$. Such a vector is said to be polarizing. By means of u we will define a partition of M^T of two subsets.

We denote

$$\mathcal{A}_p(u) := \{i \mid \alpha_{pi}(u) > 0\}, \quad \mathcal{B}_p(u) := \{k \mid \alpha_{pk}(u) < 0\}, \quad J_p(u) = \sum_e u_e J_p^e.$$

Given a number $l \in \mathbb{N}u_1 + \dots + \mathbb{N}u_r$, one defines $N_p(l, u)$ by the formula obtained from (1.1) exchanging $J(p)$ for $J_p(u)$, α_{pj} for $\alpha_{pj}(u)$, \mathcal{A}_p for $\mathcal{A}_p(u)$ and \mathcal{B}_p for $\mathcal{B}_p(u)$.

Similarly we define

$$\sigma_p(u) := (-1)^{\#\mathcal{B}_p(u)}, \quad \mathcal{Q}_\pm(u) = \{p \in M^T, \mid \sigma_p(u) = \pm 1\}.$$

Fixing a branch of the logarithmic function, then $v(\lambda) := (\lambda^{u_1}, \dots, \lambda^{u_r}) \in \mathbb{C}^r$ is a single-valued analytic function on a region \mathcal{R} of $\mathbb{C} \setminus \{0\}$. Let λ be a point of \mathcal{R} with $0 < |\lambda| < 1$, then

$$v^{-\alpha_{pj}(u)} := \prod_e (\lambda^{u_e})^{-\alpha_{pj}^e} = \lambda^{-\alpha_{pj}(u)}.$$

The right hand side of (2.7) for $t = v$ is

$$(3.7) \quad (-1)^n \sum_{p \in M^T} \sigma_p(u) \lambda^{J_p(u)} \left(\prod_{i \in \mathcal{A}_p(u)} \sum_{m=1}^{\infty} \lambda^{m\alpha_{pi}(u)} \right) \left(\prod_{k \in \mathcal{B}_p(u)} \sum_{m=0}^{\infty} \lambda^{-m\alpha_{pk}(u)} \right),$$

By (2.3) the coefficient of λ^l must be zero, for $l \in \mathbb{N}u_1 + \dots + \mathbb{N}u_r$ and $|l|$ large enough. As in the proof of Theorem 1, this coefficient

$$(-1)^n \left(\sum_{p \in \mathcal{Q}_+(u)} N_p(l, u) - \sum_{p \in \mathcal{Q}_-(u)} N_p(l, u) \right).$$

Thus we have proved the following Theorem

Theorem 5. *Let u be a vector of the Euclidean space \mathbb{R}^r which is non-orthogonal to α_{pj} for all $p \in M^T$ and all $j \in \{1, \dots, n\}$, and $l \in \mathbb{N}u_1 + \dots + \mathbb{N}u_r$ with $|l|$ sufficiently large, then*

$$\sum_{p \in \mathcal{Q}_+(u)} N_p(l, u) = \sum_{p \in \mathcal{Q}_-(u)} N_p(l, u).$$

From the proof of Theorem 5 we deduce the following Corollary

Corollary 6. *There is no an open half space in \mathbb{R}^r which contains all the α_{pj} .*

Proof. Suppose the Corollary were false. Let $0 \neq u$ be a vector in \mathbb{R}^r orthogonal to the hyperplane boundary of the half space. Then whether $\mathcal{A}_p(u)$ or $\mathcal{B}_p(u)$ would be empty for all p . Thus all the summands in (3.7) would have the same sign. This contradicts the fact that (3.7) is a finite sum. \square

As in the proof of Proposition 4, given $p \in \mathcal{Q}_+(u)$, $i \in \mathcal{A}_p(u)$ and $m \gg 1$, there exists $q_u \in \mathcal{Q}_-(u)$ such that

$$J_p(u) + m\alpha_{pi}(u) = J_{q_u}(u) + \sum_i m'_i \alpha_{q_u i}(u) - \sum_k n'_k \alpha_{q_u k}(u).$$

On the other hand, for $u' \in \mathbb{R}^r$ sufficiently close to u we have

$$\mathcal{A}_p(u) = \mathcal{A}_p(u'), \quad \mathcal{B}_p(u) = \mathcal{B}_p(u'), \quad \mathcal{Q}_\pm(u) = \mathcal{Q}_\pm(u').$$

Since $\mathcal{Q}_-(u)$ is a finite set, for u' in a neighborhood of u , the corresponding $q_{u'} \in \mathcal{Q}_-(u') = \mathcal{Q}_-(u)$ can be taken equal to q_u . Therefore we have for any u' in a neighborhood of u

$$J_p(u') + m\alpha_{pi}(u') = J_{q_u}(u') + \sum_i m'_i \alpha_{q_u i}(u') - \sum_k n'_k \alpha_{q_u k}(u').$$

That is,

$$\sum_e u'_e (J_p^e + m\alpha_{pi}^e - J_{q_u}^e - \sum_i m'_i \alpha_{q_u i}^e + \sum_k n'_k \alpha_{q_u k}^e) = 0.$$

Thus for each $e = 1, \dots, r$

$$(3.8) \quad J_p^e - J_{q_u}^e + m\alpha_{pi}^e \in I_-^e(u),$$

$I_\pm^e(u)$ being the ideal of \mathbb{Z} generated by $\{\alpha_{qj}^e \mid q \in \mathcal{Q}_\pm(u), j = 1, \dots, n\}$.

We have obtained a formula similar to (3.5). Reasoning as in Subsection 3.1 one can prove the following Theorem

Theorem 7. *Let u be a polarizing vector of the Euclidean space \mathbb{R}^r . Given $p_\pm \in \mathcal{Q}_\pm(u)$ there exists $q_\mp \in \mathcal{Q}_\mp(u)$ such that*

$$J^e(p_\pm) - J^e(q_\mp) \in I_\mp^e(u),$$

for all $e \in \{1, \dots, r\}$.

Moreover, there is a natural number c_\pm^e , with $1 \leq c_\pm^e \leq \mathcal{Q}_\mp(u)$, such that

$$c_\pm^e I_\pm^e(u) \subset I_\mp^e(u).$$

For $p \in M^T$ and $j \in \{1, \dots, n\}$ we write

$$\mathcal{R}_{pj}^\pm = \{(x_1, \dots, x_r) \in \mathbb{R}^r \mid \pm \sum_e x_e \alpha_{pj}^e > 0\}.$$

Let $t = (t_1, \dots, t_r) \in \mathbb{C}^r$, such that $(\log |t_1|, \dots, \log |t_r|) \in \mathcal{R}_{pj}^+$, then $|t^{-\alpha_{pj}}| < 1$. Hence

$$\frac{1}{1 - t^{-\alpha_{pj}}} = \sum_{m \geq 0} t^{-m\alpha_{pj}}.$$

If $(\log |t_1|, \dots, \log |t_r|) \in \mathcal{R}_{pj}^-$, then

$$\frac{1}{1 - t^{-\alpha_{pj}}} = - \sum_{m \geq 1} t^{m\alpha_{pj}}.$$

We denote by ϵ a map $M^T \times \{1, \dots, n\} \rightarrow \{\pm 1\}$, such that

$$(3.9) \quad \mathcal{K}(\epsilon) := \bigcap_{p,j} \mathcal{R}_{pj}^{\epsilon(p,j)} \neq \emptyset.$$

Since each $\mathcal{R}_{pj}^{\epsilon(p,j)}$ is an open half space of \mathbb{R}^r , if (3.9) holds, this intersection contains infinitely many points of \mathbb{R}^r as close to 0 as we wish.

We put

$$\mathcal{A}_p(\epsilon) = \{j \mid \epsilon(p, j) = -1\}, \quad \mathcal{B}_p(\epsilon) = \{j \mid \epsilon(p, j) = 1\}, \quad \sigma(p, \epsilon) = (-1)^{\#\mathcal{A}_p(\epsilon)}.$$

For any $t = (t_1, \dots, t_r) \in \mathbb{C}^r$ close to the identity of T and such that

$$(\log |t_1|, \dots, \log |t_r|) \in \mathcal{K}(\epsilon)$$

the right hand side of (2.7) is

$$(3.10) \quad \sum_{p \in M^{U(1)}} \sigma(p, \epsilon) t^{J(p)} \left(\prod_{i \in \mathcal{A}_p(\epsilon)} \sum_{m \geq 1} t^{m\alpha_{pi}} \right) \left(\prod_{k \in \mathcal{B}_p(\epsilon)} \sum_{m \geq 0} t^{-m\alpha_{pk}} \right).$$

Given $l \in \mathbb{Z}^r$ we denote by $N_p(l, \epsilon)$

$$\# \left\{ (m_1, \dots, m_a, n_1, \dots, n_b) \mid J(p) + \sum_{i \in \mathcal{A}_p(\epsilon)} m_i \alpha_{pi} - \sum_{k \in \mathcal{B}_p(\epsilon)} n_k \alpha_{pk} = l, m_i \in \mathbb{N}_{>0}, n_k \in \mathbb{N} \right\}.$$

Where $a = \#\mathcal{A}_p(\epsilon)$ and $b = \#\mathcal{B}_p(\epsilon)$. If $\#\mathcal{A}_p(\epsilon) = 0$, then $N_p(l, \epsilon)$ is the Kostant partition function corresponding to the representation of T with weights $-\alpha_{pk}$ (see [7]).

Now we define the partition $\mathcal{Q}_+(\epsilon)$, $\mathcal{Q}_-(\epsilon)$ of M^T , where

$$\mathcal{Q}_\pm(\epsilon) = \{p \in M^T \mid \sigma(p, \epsilon) = \pm 1\}.$$

As in the preceding Subsection, given $l \in \mathbb{Z}^r$ the number of times that the monomial $+t^l$ (resp. $(-1)t^l$) appears in (3.10) is

$$\sum_{p \in \mathcal{Q}_+(\epsilon)} N_p(l, \epsilon) \quad (\text{resp.} \quad \sum_{p \in \mathcal{Q}_-(\epsilon)} N_p(l, \epsilon)).$$

Hence for all $l \in \mathbb{Z}^r$ with $|l|$ sufficiently large

$$\sum_{p \in \mathcal{Q}_+(\epsilon)} N_p(l, \epsilon) = \sum_{p \in \mathcal{Q}_-(\epsilon)} N_p(l, \epsilon).$$

We have the following theorem

Theorem 8. *For any map*

$$\epsilon : M^T \times \{1, \dots, n\} \rightarrow \{\pm 1\},$$

such that $\mathcal{K}(\epsilon) \neq \emptyset$ and for any $l \in \mathbb{Z}^r$ with $|l|$ big enough,

$$\sum_{p \in \mathcal{Q}_+(\epsilon)} N_p(l, \epsilon) = \sum_{p \in \mathcal{Q}_-(\epsilon)} N_p(l, \epsilon).$$

If in the proof of Proposition 4 we substitute \mathcal{A}_p by $\mathcal{B}_p(\epsilon)$, \mathcal{Q}_+ by $\mathcal{Q}_+(\epsilon)$ and I_- by the lattice in \mathbb{R}^r

$$I_-(\epsilon) := \sum_{p \in \mathcal{Q}_-(\epsilon)} \sum_j \mathbb{Z} \alpha_{pj},$$

we obtain a proof of the following Theorem

Theorem 9. *Under the hypotheses of Theorem 8, given $p \in \mathcal{Q}_+(\epsilon)$, there exists $q \in \mathcal{Q}_-(\epsilon)$, such that $J(p) - J(q) \in I_-(\epsilon)$.*

Given $i \in \mathcal{B}_p(\epsilon)$, there exists a natural number c , $1 \leq c \leq \#\mathcal{Q}_-(\epsilon)$, such that $c\alpha_{pi} \in I_-(\epsilon)$.

4. GENERAL CASE

In the case when M^T is not a discrete set the relations which satisfy the critical values of J and the weights of the isotropy representations are more complicate. If F is a connected component of M^T , we denote by \mathcal{N}_F the normal bundle to F in M . Let

$$\oplus_{j=1}^s \mathcal{N}_{Fj}$$

be a decomposition of \mathcal{N}_F as direct sum of on line bundles, such that T acts on \mathcal{N}_{Fj} with weight $\alpha_{Fj} \in \mathbb{Z}$, with $j = 1, \dots, s$ (assumed that $s = (1/2)\text{codim } F$). The value of $\chi(t)$ is given by the following expression (see for example [11] or [17])

$$(4.1) \quad \chi(t) = \sum_F \chi_F(t)$$

where F runs on the set of components of M^T and

$$(4.2) \quad \chi_F(t) = t^{J(F)} \int_F \frac{e^\omega \text{Td}(F)}{\prod_{j=1}^s (1 - t^{-\alpha_{Fj}} e^{-c_1(\mathcal{N}_{Fj})})}$$

Here we fix a polarizing vector $u \in \mathbb{R}^r$ and we put $v(\lambda) = (\lambda^{u_1}, \dots, \lambda^{u_r}) \in \mathbb{C}^r$, as in the Subsection 3.2, where $0 < |\lambda| < 1$ is a point of the region \mathcal{R} mentioned in this Subsection. We set $\alpha_{Fj} := \sum u_e \alpha_{Fj}^e$ and $J(F) := \sum u_e J^e(F)$. The right hand side of (4.2) for $t = v$ is

$$(4.3) \quad \lambda^{J(F)} \int_F \frac{e^\omega \text{Td}(F)}{\prod_{j=1}^s (1 - \lambda^{-\alpha_{Fj}} e^{-c_1(\mathcal{N}_{Fj})})}.$$

We will write $\tau_j := \lambda^{-\alpha_{Fj}}$ and $\gamma_j := e^{-c_1(\mathcal{N}_{Fj})} - 1$. Then

$$\frac{1}{\prod_{j=1}^s (1 - \tau_j(1 + \gamma_j))} = \prod_{j=1}^s \left(\sum_{n \geq 0} \gamma_j^n \frac{\tau_j^n}{(1 - \tau_j)^{n+1}} \right).$$

So the extension of χ_F to $t = v$ is

$$(4.4) \quad \chi_F(t) = \lambda^{J(F)} \sum_{n_1, \dots, n_s \geq 0} A_{n_1 \dots n_s}(F) \frac{\tau_1^{n_1}}{(1 - \tau_1)^{n_1+1}} \cdots \frac{\tau_s^{n_s}}{(1 - \tau_s)^{n_s+1}},$$

where

$$(4.5) \quad A_{n_1 \dots n_s}(F) := \int_F e^\omega \text{Td}(F) \gamma_1^{n_1} \cdots \gamma_s^{n_s}.$$

On the other hand

$$(4.6) \quad \frac{\tau^m}{(1 - \tau)^{m+1}} = \begin{cases} \sum_{l \geq 0} C_-(m, l) \tau^l, & \text{if } |\tau| < 1 \\ (-1)^{m+1} \sum_{l \geq 0} C_+(m, l) \tau^{-l}, & \text{if } |\tau| > 1, \end{cases}$$

where

$$C_-(m, l) = \#\{(a_1, \dots, a_{m+1}) \mid a_j \in \mathbb{N}, m + a_1 + \cdots + a_{m+1} = l\}$$

$$C_+(m, l) = \#\{(a_1, \dots, a_{m+1}) \mid a_j \in \mathbb{N}, a_1 + \cdots + a_{m+1} = l - 1\}.$$

We will put $\sigma_{Fj} = \text{sign}(\alpha_{Fj})$, and $\tilde{C}_{\sigma_{Fj}}(m, l) := (-\sigma_{Fj})^{m+1} C_{\sigma_{Fj}}(m, l)$, then the fraction (4.6) when $\tau = \tau_j$ is equal to

$$\sum_{l \geq 0} \tilde{C}_{\sigma_{Fj}}(m, l) \tau_j^{-\sigma_{Fj} l}$$

If we define

$$\tilde{\alpha}_{Fj} = \begin{cases} \alpha_{Fj}, & \text{if } \alpha_{Fj} > 0 \\ -\alpha_{Fj}, & \text{if } \alpha_{Fj} < 0. \end{cases}$$

It follows from (4.4)

$$(4.7) \quad \chi_F(t) = \lambda^{J(F)} \sum_{n_1, \dots, n_s} A_{n_1, \dots, n_s} \sum_{l_1, \dots, l_s \geq 0} \tilde{C}_{\sigma_{F1}}(n_1, l_1) \dots \tilde{C}_{\sigma_{Fs}}(n_s, l_s) \lambda^{\sum_j \tilde{\alpha}_{Fj} l_j}.$$

As $A_{n_1, \dots, n_s}(F)$ vanishes if there exists j with $n_j > (1/2)\dim M$, the first sum in (4.7) can be restricted to $\vec{n} = (n_1, \dots, n_{s_F}) \in \mathcal{G}(F)$, where $s_F := (1/2)\text{codim } F$,

$$\mathcal{G}(F) = \{\vec{n} \in \overbrace{G \times \dots \times G}^{s_F}\},$$

and $G := \{1, \dots, n\}$. The coefficient of λ^k in $\chi(t)$ is

$$\sum_F \left(\sum_{\vec{n} \in \mathcal{G}(F)} A_{\vec{n}}(F) \sum_{l \in \mathcal{L}(F, k)} \left(\prod_{j=1}^{s_F} \tilde{C}_{\sigma_{Fj}}(n_j, l_j) \right) \right),$$

where

$$\mathcal{L}(F, k) = \{(l_1, \dots, l_{s_F}) \mid l_j \in \mathbb{N}, \sum_j \tilde{\alpha}_{Fj} l_j = k - J(F)\}.$$

For k large enough the coefficient of λ^k in $\chi(t)$ is zero. Thus we have proved the following Theorem

Theorem 10. *For any polarizing vector $u \in \mathbb{R}^r$ and $k \in \mathbb{N}u_1 + \dots + \mathbb{N}u_r$ with $|k|$ sufficiently large*

$$(4.8) \quad \sum_F \left(\sum_{\vec{n} \in \mathcal{G}(F)} A_{\vec{n}}(F) \sum_{l \in \mathcal{L}(F, k)} \left(\prod_{j=1}^{s_F} \tilde{C}_{\sigma_{Fj}}(n_j, l_j) \right) \right) = 0.$$

We denote by $D(F, \vec{n}, k)$ for the combinatorial number

$$\sum_{l \in \mathcal{L}(F, k)} \left(\prod_{j=1}^{s_F} C_{\sigma_{Fj}}(n_j, l_j) \right).$$

Let

$$\tau(F, \vec{n}) := \prod_{j=1}^{s_F} (-\sigma_{Fj})^{n_j+1}$$

and

$$\mathcal{H} := \{(F, \vec{n}) \mid F \in \mathcal{F}, \vec{n} \in \mathcal{G}(F)\}.$$

On \mathcal{H} we have the map $\tau : (F, \vec{n}) \in \mathcal{H} \mapsto \tau(F, \vec{n}) \in \{\pm 1\}$. One has the obvious partition \mathcal{H}^+ , \mathcal{H}^- for \mathcal{H} . From Theorem 10 we deduce the following Corollary whose statement has the form of that of Theorem 8.

Corollary 11. *For any $k \in \mathbb{N}u_1 + \dots + \mathbb{N}u_r$ with $|k|$ sufficiently large and any generic vector $u \in \mathbb{R}^r$*

$$\sum_{(F, \vec{n}) \in \mathcal{H}^+} A(F, \vec{n}) D(F, \vec{n}, k) = \sum_{(F, \vec{n}) \in \mathcal{H}^-} A(F, \vec{n}) D(F, \vec{n}, k),$$

where $A(F, \vec{n})$ is given by (4.5).

Remark. If M^T is a set of isolated points and $p \in M^T$, then $A_{0\dots 0}(p) = 1$ and the others $A_{n_1\dots n_s}(p)$ vanish. Since $C_-(0, l) = 1$, $C_+(0, l) = 1$ for $l > 0$ and $C_+(0, 0) = 0$ the number of nonzero summands in

$$\sum_{l \in \mathcal{L}(p, k)} \left(\prod_{j=1}^n \tilde{C}_{\sigma_{pj}}(0, l_j) \right)$$

is precisely the number $N_p(k, u)$ introduced in Subsection 3.2. Thus we deduce from (4.8)

$$\sum_{p \in Q_+(u)} N_p(k, u) - \sum_{p \in Q_-(u)} N_p(k, u) = 0,$$

which agrees with Theorem 5.

Example 2.

Let us suppose that $\dim M = 4$ and it is acted on by S^1 so that M^{S^1} has two connected components: A point q and F , a 2-submanifold of M . Let us assume that $J(q)$ is the minimum value of J and $J(F)$ the maximum one. So $\alpha_{qj} > 0$ for $j = 1, 2$ and $\alpha_F < 0$.

It is easy to determine the contribution of q to (4.8) since the only nonzero $A_{n_1 n_2}(q)$ is $A_{00}(q) = 1$. This contribution is

$$\sharp\{(l_1, l_2) \mid l_j \in \mathbb{N}_{>0}, \alpha_{q1}l_1 + \alpha_{q2}l_2 = k - J(q)\}.$$

On the other hand $A_0(F) = \int_F (\omega + \text{Td}_1(F))$ and $A_1(F) = -\int_F c_1(\mathcal{N}_F)$. The other $A_n(F)$ are equal to zero. Hence

$$\chi(t) = \sum_{m_1, m_2 \geq 1} t^{m_1 \alpha_{q1} + m_2 \alpha_{q2} + J(q)} + A_0 \sum_{l \geq 0} t^{J(F) - l \alpha_F} + A_1 \sum_{l_1, l_2 \geq 0} t^{J(F) - (l_1 + l_2 + 1) \alpha_F}.$$

For any $m_1 \gg 1$ the exponent of the monomial $t^{m_1 \alpha_{q1} + m_2 \alpha_{q2} + J(q)}$ must appear in a monomial of type $t^{J(F) - m \alpha_F}$. Therefore $J(F) - J(q) \in I_-$, where I_- is the ideal of \mathbb{Z} generated by α_F .

Given $k \in \mathbb{N}$, if

$$(4.9) \quad n_0 := \frac{J(F) - k}{\alpha_F} \notin \mathbb{N},$$

the coefficient of t^k in $\chi(t)$ is

$$N_q(k) = \sharp\{(m_1, m_2) \mid m_j > 0, m_1 \alpha_{q1} + m_2 \alpha_{q2} + J(q) = k\} = 0.$$

Thus for k sufficiently large, $N_q(k) = 0$ (assumed that (4.9) holds).

If $n_0 := (J(F) - k)/\alpha_F \in \mathbb{N}$, the coefficient of t^k in $\chi(t)$ is $N_q(k) + A_0 + n_0 A_1$, since

$$\sharp\{(l_1, l_2) \mid l_j \in \mathbb{N}, J(F) - (l_1 + l_2 + 1) \alpha_F = k\} = n_0.$$

So we can state the following Proposition.

Proposition 12. *Let M be a 4-manifold acted on by S^1 . If the fixed point set has only two components, q and F of dimensions 0 and 2 respectively and $J(F) > J(q)$, then*

1) *There is a natural number m such mapping*

$$n_0 \in \mathbb{N}_{>m} \mapsto \sharp\{(m_1, m_2) \mid m_j > 0, m_1 \alpha_{q1} + m_2 \alpha_{q2} = J(F) - J(q) - n_0 \alpha_F\}$$

is the affine map $-A_0(F) - n_0 A_1(F)$.

2)

$$\sharp\{(m_1, m_2) \mid m_j > 0, m_1\alpha_{q1} + m_2\alpha_{q2} = k - J(q)\} = 0$$

for all k sufficiently large such that $(J(F) - k)/\alpha_F \notin \mathbb{N}$.

3) $J(F) - J(q) \in I_-$, where I_- is the ideal of \mathbb{Z} generated by α_F .

Now we return to the toric manifold M_Δ considered in *Example 1* (Subsection (3.1)).

$$M_\Delta = \{(z_0, z_1, z_2) \mid \sum_j |z_j|^2 = 1\} / \sim.$$

The $T = (U(1))^2$ -action on M_Δ is defined by (see [7])

$$(\lambda_0, \lambda_1)[z] = [\lambda_0 z_0, \lambda_1 z_1, z_2].$$

We write $z_j = \rho_j \exp(i\theta_j)$, with $\theta_j \in \mathbb{R}/\mathbb{Z}$. Then $(\rho_0, \varphi_0, \rho_1, \varphi_1)$, with $\varphi_j = \theta_j - \theta_2$, are coordinates on M_Δ , and the map $J_T([z]) = (\rho_0^2, \rho_1^2)$ satisfies $\text{im } J_T = \Delta$. If $Y = (ai, bi) \in \mathfrak{t}$, then $Y_M = -a \frac{\partial}{\partial \varphi_0} - b \frac{\partial}{\partial \varphi_1}$. Our convention $\omega(Y_M, \cdot) = d\langle \mu, Y \rangle = \text{id}\langle J_T, Y \rangle$ imposes that

$$\omega = -(d\rho_0^2 \wedge d\varphi_0 + d\rho_1^2 \wedge d\varphi_1).$$

Next we consider the S^1 -action on M_Δ generated by $X = (x, x) \in \mathbb{Z}^2$. The components of the fixed point set for this S^1 -action are $p = [0, 0, 1]$ and $F = \{[z] \mid z_2 = 0\} \simeq \mathbb{CP}^1$. The weights of the isotropy representations are

$$\alpha_{p1} = x, \alpha_{p2} = x, \alpha_F = -x, \text{ and } J(p) = 0, J(F) = x.$$

The tangent bundle TF is $\mathcal{O}(2)$ and the normal bundle of \mathbb{CP}^1 in \mathbb{CP}^2 is $\mathcal{O}(1)$. Moreover

$$\int_F \omega = - \int_0^1 d\rho_0^2 \int_0^1 d\varphi_0 = -1.$$

Thus $A_0 = 0$, $A_1 = -1$ and $-A_0(F) - n_0 A_1(F) = n_0$. Furthermore $\sharp\{(m_1, m_2) \mid m_j > 0, m_1 + m_2 = 1 + n_0\} = n_0$, in accordance with the first item of Proposition 12.

REFERENCES

- [1] Atiyah, M.F.: Convexity and commuting Hamiltonians. *Bull. London Math. Soc.* **14**, 1-15 (1982).
- [2] Atiyah, M. F., Segal G.B: The index of elliptic operators II. *Ann. of Math* **87**, 531-545 (1968).
- [3] Berline, N., Getzler, E., Vergne, M.: *Heat kernels and Dirac operators*. Springer-Verlag, Berlin, 1991.
- [4] Brion, M., Vergne, M.: Residue formulae, vector partition functions and lattice points in rational polytopes. *J. Amer. Math. Soc.* **10**, 797-833 (1997).
- [5] Duistermaat, H., Guillemin, V., Meinrenken, E., Wu, S. : Symplectic reduction and Riemann-Roch for circle actions. *Math. Res. Lett.* **2**, 259-266 (1995).
- [6] Guillemin, V.: Reduced phase spaces and Riemann-Roch. *Lie theory and geometry*, 305-334, *Progr. Math.*, 123, Birkhuser Boston, Boston, MA, 1994.
- [7] Guillemin, V.: *Moment maps and combinatorial invariants of Hamiltonian T^n -spaces*. Birkhäuser, Boston, 1994.
- [8] Guillemin, V.: Riemann-Roch for toric orbifolds *J. Differential Geom.* **45**, 53-73 (1997).
- [9] Guillemin, V., Sternberg, S.: Convexity properties of the moment mapping. *Invent. Math.* **67**, 491-513 (1982).
- [10] Guillemin, V., Sternberg, S.: Geometric quantization and multiplicities of group representations. *Invent. Math.* **67**, 515-538 (1982).
- [11] Jeffrey, L.C., Kirwan, F.C.: On localization and Riemann-Roch numbers for symplectic quotients. *Quart. J. Math. Oxford Ser. (2)* **47**, no. 186, 165-185 (1996).

- [12] Kirwan, F.: Convexity properties of the moment mapping, III. *Invent. math.* **77**, 547-552 (1984).
- [13] Lawson, H. B., Michelsohn, M.-L. : *Spin geometry*. Princeton University Press, Princeton, 1989.
- [14] McDuff, D., Salamon, D.: *Introduction to symplectic topology*. Clarenton Press, Oxford, 1998.
- [15] Meinrenken, E.: Symplectic surgery and Spin^c -Dirac operator. *Adv. Math.* **134**, 240-277 (1998).
- [16] Polterovich, L.: *The geometry of the group of symplectic diffeomorphisms*. Birkhäuser, Basel, 2001.
- [17] Sjamar, R.: Symplectic reduction and Riemann-Roch formulas for multiplicities. *Bull. Amer. Math. Soc. (N.S.)* **33**, 327-338, (1996).
- [18] Vergne, M.: Multiplicities formulas for geometric quatization. Part I, II. *Duke Math. J.* **82** 143-194 (1996).
- [19] Viña, A.: Symplectic action around loops in $\text{Ham}(M)$. *Geom. Dedicata* **109**, 31-49, (2004).
- [20] Weinstein, A.: Cohomology of symplectomorphism groups and critical values of Hamiltonians. *Math. Z.* **201**, 75-82 (1989)
- [21] Woodhouse, N.M.J. *Geometric quantization*. Clarenton Press, Oxford, 1992.

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